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Discrete time evolution and energy nonconservation in noncommutative physics

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Abstract: Time-space noncommutativity leads to quantisation of time [1] and energy nonconservation [2] when time is conjugate to a compact spatial direction like a circle. In this context energy is conserved only modulo some fixed unit. Such a possibility arises for example in theories with a compact extra dimension with which time does not commute. The above results suggest striking phenomenological consequences in extra dimensional theories and elsewhere. In this paper we develop scattering theory for discrete time translations. It enables the calculation of transition probabilities for energy nonconserving processes and has a central role both in formal theory and phenomenology.
We can also consider space-space noncommutativity where one of the spatial directions is a circle. That leads to the quantisation of the remaining spatial direction and conservation of momentum in that direction only modulo some fixed unit, as a simple adaptation of the results in this paper shows.

Keywords: Non-Commutative Geometry, Field Theories in Lower Dimensions.

Contents

1. Introduction

In most approaches to quantum theory *time* is a real parameter occurring in the evolution operator. On the other hand, spatial coordinates can be seen as operators acting on a Hilbert space of states.

Many authors have recently considered quantum field theories defined on noncommutative space-times. In this context time is another coordinate, and together with the other coordinates it generates a noncommutative C^*-algebra. Ideas coming from noncommutative geometry have been extensively used in this research program.

In one of the most studied models, the noncommutative space-time $\mathbb{R}^D_\theta = \mathcal{A}_\theta(\mathbb{R}^D)$ is described by a deformation of the algebra of functions over $\mathbb{R}^D$, the deformed product being the Groenewold-Moyal (GM) product between complex-valued functions. It has been claimed that due to the nonlocal character of the GM product, QFT's constructed in this way have intrinsic unitarity problems [3]. Some authors have tried to avoid this problem reformulating the rules leading to the scattering amplitudes [4]. The authors of [5] have shown how to construct unitary quantum field theories using noncommutative coordinates of the canonical (GM) type. Our approach to quantum physics started in [2, 6] is inspired by [5].

In [2], we studied a class of noncommutative space-times where time evolution is discrete. The simplest example is the noncommutative cylinder $\mathcal{A}_\theta(\mathbb{R}\times S^1)$. We describe this space as the noncommutative unital C^*-algebra generated by $\hat{x}_0$ and $e^{i\hat{x}_1}$, with the defining relation:

$$e^{i\hat{x}_1}\hat{x}_0 = \hat{x}_0 e^{i\hat{x}_1} + \theta e^{i\hat{x}_1} \, . \tag{1.1}$$

This can be seen as naturally coming from the canonical commutation relation $[\hat{x}_0, \hat{x}_1] = i\theta\mathbb{I}$ defining the GM plane $\mathbb{R}^2_\theta = \mathcal{A}_\theta(\mathbb{R}^2)$ when restricted to periodic functions of $\hat{x}_1$.

The relation (1.1) has two direct consequences: the spectrum of the time coordinate $\hat{x}_0$ is discrete [1], being spaced by the fundamental time scale θ, and more interestingly, the time evolution is given by integral powers of the evolution operator corresponding to a time interval equal to θ. This fact leads to the possibility of energy nonconservation in scattering processes [2]: it is conserved only modulo $\frac{2\pi}{\theta}$, as we shall see in this paper. (See also [7 − 9] in this connection.)

It would be natural to think of $\mathcal{A}_\theta(\mathbb{R}\times S^1)$ as a two-dimensional noncommutative subspace of a larger space-time, the noncommuting coordinate being interpreted as a warped space-like extra dimension. In this picture, discretisation of time translation and attendant energy nonconservation would have significant phenomenological consequences which wait to be explored. This space-time picture is appropriate from a field-theoretical point of view as well. But in this paper we are concerned about quantum mechanics, so that we can also interpret the noncommuting variable as a sort of extra degree of freedom, which does not commute with the time coordinate.

Let $Q \times S^1$ be the configuration space of a quantum system, Q being an ordinary D-dimensional configuration space. Consider the algebra $\mathcal{A}$ generated by $\hat{q}_1 \cdots, \hat{q}_D$, $e^{i\hat{x}_1}$ and $\hat{x}_0$. We have:

$$\mathcal{A} = \mathcal{F}(Q) \otimes \mathcal{A}_\theta\left(\mathbb{R}\times S^1\right) \quad , \quad \hat{q}_i e^{i\hat{x}_1} = e^{i\hat{x}_1}\hat{q}_i \, , \tag{1.2}$$

where $\mathcal{F}(Q)$ is the commutative algebra of functions over the configuration space Q, generated by the coordinates (coordinate functions) $\hat{q}_1, \cdots, \hat{q}_D$. The only noncommutative piece of $\mathcal{A}$ is the noncommutative cylinder.

The algebra $\mathcal{A}$ is the noncommutative analogue of the space of (time dependent) wave functions of ordinary quantum theory. We point out that the operators $\hat{q}_1, \cdots, \hat{q}_D$ and $e^{i\hat{x}_1}$ play the role of "coordinates". Just like in the usual case, we can define position and momentum operators acting on elements of $\mathcal{A}$.

The position operators associated to Q are the same as in ordinary quantum theory. They are denoted by $\hat{q}^L_1, \cdots, \hat{q}^L_D$, the superscript ^{L} meaning left multiplication. We can also introduce the multiplication operator $e^{i\hat{x}^L_1}$. Let $\hat{p}_i$ be the momentum canonically conjugate to $\hat{q}^L_i$ and let $\hat{P}_1$ be the noncommutative analogue of $-i\partial_1$, defined by

$$\hat{P}_1 e^{i\hat{x}_1} = e^{i\hat{x}_1} \, . \tag{1.3}$$

If $\hat{H}$ is the Hamiltonian operator describing a quantum system based on $\mathcal{A}$, we can write it generically as:

$$\hat{H} = \hat{H}\left(\hat{q}^L_1, \cdots, \hat{q}^L_D, \hat{p}_1, \cdots, \hat{p}_D, e^{i\hat{x}^L_1}, \hat{P}_1\right) \, . \tag{1.4}$$

As we shall see later, it is the dependence of $\hat{H}$ on $e^{i\hat{x}^L_1}$ and $\hat{P}_1$ that causes the discretization of time translation and time evolution controlled by $\hat{H}$.

A variant of the noncommutative spacetime model considered here would be one where two spatial directions do not commute and one of them is a circle. Then the remaining spatial direction, which we can identify with $\hat{x}_0$ in (1.1), gets quantised in units of θ and

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momentum in that direction is conserved only modulo $\frac{2\pi}{\theta}$. The proofs of these results also follow from the considerations of this paper.

2. The algebra

In this section we study the algebraic structure of $\mathcal{A}$. Its noncommutative piece is responsible for all its interesting aspects, so let us recall some facts about the noncommutative cylinder $\mathcal{A}_\theta(\mathbb{R}\times S^1)$. It follows from (1.1) that

$$e^{i\frac{2\pi}{\theta}\hat{x}_0}e^{i\hat{x}_1}=e^{i\hat{x}_1}e^{i\frac{2\pi}{\theta}\hat{x}_0}\,. \tag{2.1}$$

Hence in every unitary irreducible representation of $\mathcal{A}_\theta(\mathbb{R}\times S^1)$, the central element $e^{i\frac{2\pi}{\theta}\hat{x}_0}$ is just a phase:

$$e^{i\frac{2\pi}{\theta}\hat{x}_0}=e^{i\varphi}\mathbb{I}\,. \tag{2.2}$$

For the spectrum $\operatorname{spec}\hat{x}_0$ of $\hat{x}_0$ in such a representation, we have from (2.2),

$$\operatorname{spec}\hat{x}_0=\theta\left(\mathbb{Z}+\frac{\varphi}{2\pi}\right)\equiv\left\{\theta\left(n+\frac{\varphi}{2\pi}\right):n\in\mathbb{Z}\right\}\,, \tag{2.3}$$

which leads to the important relation:

$$e^{i\left(\omega+\frac{2\pi}{\theta}\right)\hat{x}_0}=e^{i\varphi}\,e^{i\omega\hat{x}_0}\,. \tag{2.4}$$

We can realize $\mathcal{A}_\theta(\mathbb{R}\times S^1)$ irreducibly on an auxiliary Hilbert space $L^2(S^1)$. On this space, $e^{i\hat{x}_1}$ acts like a multiplication operator,

$$\left(e^{i\hat{x}_1}\alpha\right)\left(e^{ix_1}\right)=e^{ix_1}\,\alpha\left(e^{ix_1}\right)\,, \tag{2.5}$$

while $-\hat{x}_0/\theta$ acts like an ordinary momentum operator,

$$\left(\hat{x}_0\alpha\right)\left(e^{ix_1}\right)=i\theta\frac{\partial\alpha}{\partial x_1}\left(e^{ix_1}\right)\,. \tag{2.6}$$

We denote this representation as $\mathcal{A}_\theta(\mathbb{R}\times S^1,e^{i\frac{\varphi}{2\pi}})$. The flux term $\frac{\varphi}{2\pi}$ classifies all unitary irreducible representations of $\mathcal{A}_\theta(\mathbb{R}\times S^1)$, every such representation being equivalent to $\mathcal{A}_\theta(\mathbb{R}\times S^1,e^{i\frac{\varphi}{2\pi}})$ for some value of φ.

In what follows we deal directly with $\mathcal{A}_\theta(\mathbb{R}\times S^1,e^{i\frac{\varphi}{2\pi}})$, instead of the algebra $\mathcal{A}_\theta(\mathbb{R}\times S^1)$ itself. In order to do this, we need an explicit description of elements of $\mathcal{A}_\theta(\mathbb{R}\times S^1,e^{i\frac{\varphi}{2\pi}})$. The first thing we note is that only integral powers of $e^{i\hat{x}_1}$ belong to this representation. In fact, it follows from (1.1) that

$$e^{i\alpha\hat{x}_1}e^{i\lambda\hat{x}_0}e^{-i\alpha\hat{x}_1}=e^{i\lambda(\hat{x}_0+\alpha\theta)}\,. \tag{2.7}$$

Conjugating the relation (2.4) by $e^{i\alpha\hat{x}_1}$ and using (2.2) and (2.7), we get

$$e^{i2\pi\alpha}=1\quad\text{or}\quad\alpha\in\mathbb{Z}\,. \tag{2.8}$$

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Now consider an element of the subalgebra generated by $\hat{x}_0$. We can write it generically as

$$\hat{\alpha} = \int_{-\infty}^{\infty} d\omega\, \tilde{\alpha}(\omega) e^{i\omega\hat{x}_0} \,. \tag{2.9}$$

Let us split this integral into an infinite sum of integrals, each one taken over a finite interval of length $\frac{2\pi}{\theta}$,

$$\begin{aligned}\hat{\alpha} = \sum_{m\in\mathbb{Z}} \int_{(2m-1)\frac{\pi}{\theta}}^{(2m+1)\frac{\pi}{\theta}} d\omega\tilde{\alpha}\left(\omega\right) e^{i\omega\hat{x}_0} &= \sum_{m\in\mathbb{Z}} \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\tilde{\alpha}\left(\omega + m\frac{2\pi}{\theta}\right) e^{im\frac{2\pi}{\theta}\hat{x}_0} e^{i\omega\hat{x}_0} \\ &= \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega \sum_{m\in\mathbb{Z}} \tilde{\alpha}\left(\omega + m\frac{2\pi}{\theta}\right) e^{im\varphi}\, e^{i\omega\hat{x}_0} \,, \end{aligned} \tag{2.10}$$

where we have made use of (2.2). Notice that the coefficient

$$f_{\tilde{\alpha}}\left(\omega\right) := \sum_{m\in\mathbb{Z}} \tilde{\alpha}\left(\omega + m\frac{2\pi}{\theta}\right) e^{im\varphi}$$

in (2.10) is quasi-periodic:

$$f_{\tilde{\alpha}}\left(\omega + \frac{2\pi}{\theta}\right) = e^{-i\varphi} f_{\tilde{\alpha}}(\omega) \,. \tag{2.11}$$

From this we conclude that the most general element of $\mathcal{A}_\theta(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}})$ can be written as

$$\hat{\psi} = \sum_{n\in\mathbb{Z}} \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\, \tilde{\psi}_n(\omega)\, e^{in\hat{x}_1} e^{i\omega\hat{x}_0} \,, \tag{2.12}$$

where $\tilde{\psi}_n(\omega + \frac{2\pi}{\theta}) = e^{-i\varphi}\tilde{\psi}_n(\omega)$.

Taking (1.2) into account we write the following expression for the generic element of $\tilde{\mathcal{A}} := \mathcal{F}(Q) \otimes \mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right)$:[1]

$$\hat{\psi} = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} d^D k \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\, \tilde{\psi}_n(\vec{k},\omega) e^{i\sum_i k_i\hat{q}_i}\, e^{in\hat{x}_1} e^{i\omega\hat{x}_0} \,, \tag{2.13}$$

where $\tilde{\psi}_n(\vec{k}, \omega + \frac{2\pi}{\theta}) = e^{-i\varphi}\tilde{\psi}_n(\vec{k},\omega)$.

2.1 Translation automorphisms

In analogy with the common practice in quantum mechanics, wave functions will be written in the "basis" $\{e^{i\sum_i k_i\hat{q}_i}, e^{in\hat{x}_1}\}$, just like in (2.13). In order to proceed with the construction of a noncommutative quantum theory, we have to study the action of time and space translations on $\tilde{\mathcal{A}}$. The product structure (1.2) shows that the momenta of $\mathcal{F}(Q)$ act on $\tilde{\mathcal{A}}$ as in the ordinary case. It remains to investigate the action of the momenta $\hat{P}_0$ and $\hat{P}_1$, which act on the GM plane $\mathbb{R}^2_\theta = \mathcal{A}_\theta(\mathbb{R}^2)$ as:

$$\hat{P}_0 = -\frac{1}{\theta}[\hat{x}_1\,,\cdot] \quad , \quad \hat{P}_1 = -\frac{1}{\theta}[\hat{x}_0\,,\cdot]\,, \tag{2.14}$$

[1] Notice that $\tilde{\mathcal{A}}$ comes from $\mathcal{A}$ when we replace $\mathcal{A}_\theta(\mathbb{R}\times S^1)$ by $\mathcal{A}_\theta(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}})$.

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so that

$$\hat{P}_\mu \hat{x}_\nu = i\,\eta_{\mu\nu}\mathbb{I}\,, \tag{2.15}$$

where $\eta_{\mu\nu} = \mathrm{diag}\,[1,-1]$.

Using the Jacobi identity (fulfilled by the commutator), one can show that the momenta $\hat{P}_\mu$ generate a commutative algebra. $\hat{P}_\mu$ are quite similar to their usual counterparts $i\partial_0$ and $-i\partial_1$.

The only restriction on the action of $\hat{P}_0$ on $\tilde{\mathcal{A}}$ comes from the quasi-periodicity of the coefficients $\tilde{\psi}_n(\vec{k},\omega)$ in (2.13). In fact, let us calculate the action of an arbitrary time translation on (2.13):

$$\begin{aligned} e^{-i\tau\hat{P}_0}\hat{\psi} &= \sum_{n\in\mathbb{Z}}\int_{-\infty}^{+\infty} d^Dk \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\,\tilde{\psi}_n(\vec{k},\omega) e^{i\sum_i k_i\hat{q}_i}\, e^{in\hat{x}_1} e^{i\omega(\hat{x}_0+\tau\mathbb{I})} \\ &= \sum_{n\in\mathbb{Z}}\int_{-\infty}^{+\infty} d^Dk \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\,\tilde{\psi}_n(\vec{k},\omega) e^{i\omega\tau} e^{i\sum_i k_i\hat{q}_i}\, e^{in\hat{x}_1} e^{i\omega\hat{x}_0}\,. \end{aligned} \tag{2.16}$$

If $e^{-i\tau\hat{P}_0}$ is to act on $\tilde{\mathcal{A}}$, then the new coefficients $\tilde{\psi}_n(\vec{k},\omega)e^{i\omega\tau}$ must also be quasi-periodic functions of ω. This condition is only fulfilled if $\tau\in\theta\mathbb{Z}$. This shows that the time translations on $\tilde{\mathcal{A}}$ are powers of a minimum translation $e^{-i\theta\hat{P}_0}\hat{x}_0 = \hat{x}_0 + \theta\mathbb{I}$.

The spatial translations (generated by $\hat{P}_1$) act on $\tilde{\mathcal{A}}$ without any restriction.

3. Dynamics and Hilbert spaces

In this section we recall the construction of Hilbert spaces in the present approach to quantum physics. Details can be found in [6], where the relation to the GNS construction is also discussed.

3.1 Symbols and positive functionals

The first step is the introduction of a linear application from $\tilde{\mathcal{A}}$ to the space of complex functions. We associate to every $\hat{\psi}$ as in (2.13) the function $\psi : \mathbb{R}^D \times S^1 \times \theta\left(\mathbb{Z}+\frac{\varphi}{2\pi}\right) \longrightarrow \mathbb{C}$ given by:

$$\psi\left(\vec{q}, e^{ix_1}, \theta\left(m+\frac{\varphi}{2\pi}\right)\right) = \sum_{n\in\mathbb{Z}}\int_{-\infty}^{+\infty} d^Dk \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\,\tilde{\psi}_n(\vec{k},\omega) e^{i\sum_i k_i q_i}\, e^{inx_1} e^{i\omega\theta\left(m+\frac{\varphi}{2\pi}\right)}\,. \tag{3.1}$$

Equation (3.1) defines the *symbol* ψ of the operator $\hat{\psi}$.

Now we define a family of positive linear functionals on $\tilde{\mathcal{A}}$:

$$S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}(\hat{\psi}) = \int_{-\infty}^{+\infty} d^Dq \int_0^{2\pi} dx_1\, \psi\left(\vec{q}, e^{ix_1}, \theta\left(m+\frac{\varphi}{2\pi}\right)\right)\,. \tag{3.2}$$

In order to verify that (3.2) is a positive functional, one first checks that

$$S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}(\hat{\psi}^*\hat{\phi}) = \int_{-\infty}^{+\infty} d^Dq \int_0^{2\pi} dx_1 \overline{\psi\left(\vec{q}, e^{ix_1}, \theta\left(m+\frac{\varphi}{2\pi}\right)\right)}\,\phi\left(\vec{q}, e^{ix_1}, \theta\left(m+\frac{\varphi}{2\pi}\right)\right)\,, \tag{3.3}$$

which shows that $S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}(\hat{\psi}^*\hat{\psi}) \geq 0$.

It is natural to try to define an inner product in the following way:

$$\left(\hat{\psi}, \hat{\phi}\right)_{\theta\left(m+\frac{\varphi}{2\pi}\right)} = S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}(\hat{\psi}^*\hat{\phi}) \,. \tag{3.4}$$

This family of sesquilinear forms have the right linearity properties, but fails to give $\tilde{\mathcal{A}}$ a Hilbert space structure. For any given value of m we can find nontrivial null vectors. In fact, let ψ be the symbol of $\hat{\psi} \in \tilde{\mathcal{A}}$. If $\psi\left(\vec{q}, e^{ix_1}, \theta\left(m + \frac{\varphi}{2\pi}\right)\right) = 0$ for some *fixed* $m \in \mathbb{Z}$ and for *all* $(\vec{q}, e^{ix_1}) \in \mathbb{R}^D \times S^1$, then it follows from (3.3) that:

$$S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}(\hat{\psi}^*\hat{\psi}) = 0 \,. \tag{3.5}$$

But $\psi\left(\cdot\,, \cdot\,, \theta\left(n + \frac{\varphi}{2\pi}\right)\right)$ need not be zero for $n \neq m$ and hence $\hat{\psi}$ need not be zero. Such nonzero $\hat{\psi}$ are nontrivial null vectors for this form.

Besides this feature, if $n \neq m$ the forms labeled by $\theta\left(n + \frac{\varphi}{2\pi}\right)$ and $\theta\left(m + \frac{\varphi}{2\pi}\right)$ are not directly related to each other. This time-dependence of the "inner product" is not a desirable feature in an approach to quantum theory which is to reduce to ordinary quantum mechanics in the appropriate commutative limit.

3.2 The Hilbert space of states

The solution to the two problems mentioned in the last section consists in finding a subspace $\mathcal{H}_\theta$ of $\tilde{\mathcal{A}}$ such that: (i) the only null vector in $\mathcal{H}_\theta$ is 0 and (ii) the family of positive sesquilinear forms (3.4) collapses to a unique true inner product.

Recall that in usual quantum mechanics, the Hilbert space of states is the space of square integrable functions over the configuration space Q ($\mathcal{H} := L^2(Q)$). Let $\psi \in \mathcal{H}$, so that $\psi(q) \in \mathbb{C}$, for all $q \in Q$. Time translation applied to ψ is equivalent to time evolution, and is given by the action of the evolution operator $e^{-i\tau H}$ on ψ, where H is the Hamiltonian operator. As H is self-adjoint, the evolution operator is unitary, thereby defining an isometry in $\mathcal{H} = L^2(Q)$. Hence we see that the evolved wave function $\psi(\cdot\,, \tau)$ belongs to the same Hilbert space as $\psi(\cdot)$.

In the noncommutative case, time translation is given by the action of the translation operator $U(N\theta) := e^{-iN\theta\hat{P}_0}$ on elements of $\tilde{\mathcal{A}}$. One can check that $e^{-iN\theta\hat{P}_0}$ is not a unitary operator on $\tilde{\mathcal{A}}$ (for the sesquilinear form (3.4)). In fact, there are pairs $(\hat{\psi}\,, \hat{\phi}) \in \tilde{\mathcal{A}} \times \tilde{\mathcal{A}}$ such that:

$$\left(e^{-iN\theta\hat{P}_0}\hat{\psi}\,, \hat{\phi}\right)_{\theta\left(m+\frac{\varphi}{2\pi}\right)} \neq \left(\hat{\psi}\,, e^{-iN\theta\hat{P}_0}\hat{\phi}\right)_{\theta\left(m+\frac{\varphi}{2\pi}\right)} . \tag{3.6}$$

Hence time translation $e^{-iN\theta\hat{P}_0}$ cannot be used to identify different metric spaces $\left(\tilde{\mathcal{A}}\,, (\cdot\,, \cdot)_{\left(m+\frac{\varphi}{2\pi}\right)}\right)$ labelled by m unless there is a suitable constraint. Such a constraint is inspired by standard quantum mechanics. We next explain it. It is a generalization of Schrödinger's equation.

Consider a time-independent Hamiltonian $\hat{H}$ (this means that $[\hat{P}_0, \hat{H}] = 0$), written generically as:

$$\hat{H} = \hat{H}(\vec{q}, e^{i\hat{x}_1}, \vec{p}, \hat{P}_1) \,, \tag{3.7}$$

where $\vec{q} = (\hat{q}_1^L, \cdots, \hat{q}_D^L)$, $\vec{p} = (\hat{p}_1, \cdots, \hat{p}_D)$ and $\hat{P}_1$ is the momentum operator defined in (2.15).

We suppose that (3.7) is *hermitian* in the following sense:

$$\left(\hat{\psi}, \hat{H}\hat{\phi}\right)_{\theta\left(m+\frac{\varphi}{2\pi}\right)} = \left(\hat{H}\hat{\psi}, \hat{\phi}\right)_{\theta\left(m+\frac{\varphi}{2\pi}\right)} . \tag{3.8}$$

Now in [6], to resolve a similar problem, the Hilbert spaces were defined as subspaces of the full noncommutative algebra ($\mathcal{A}_\theta(\mathbb{R}^{D+1})$ in that case) fulfilling an algebraic condition, the noncommutative analogue of Schrödinger's equation. Here we cannot do this, because the generator of time translation $\hat{P}_0$ does not act on $\tilde{\mathcal{A}}$. But we showed in section 2 that integral powers of $e^{-i\theta\hat{P}_0}$ do act on $\tilde{\mathcal{A}}$, so that we can impose the following substitute for the Schrödinger's constraint:

$$e^{-i\theta\hat{P}_0}\hat{\psi} = e^{-i\theta\hat{H}}\hat{\psi} . \tag{3.9}$$

Let $\mathcal{H}_\theta$ denote the subspace of $\tilde{\mathcal{A}}$ consisting only of solutions of (3.9):

$$\mathcal{H}_\theta := \left\{\hat{\psi} \in \tilde{\mathcal{A}} \,:\, e^{-i\theta\hat{P}_0}\hat{\psi} = e^{-i\theta\hat{H}}\hat{\psi}\right\} . \tag{3.10}$$

The elements of $\mathcal{H}_\theta$ are of the form:

$$\hat{\psi} = e^{-i\hat{x}_0^R\hat{H}}\,\hat{\chi}\left(\vec{q}, e^{i\hat{x}_1}\right) , \tag{3.11}$$

where $\hat{x}_0^R$ is multiplication by $\hat{x}_0$ from the right.

We show below that the space $\mathcal{H}_\theta$ just defined is a true Hilbert space, with inner product given by (3.4).

Suppose $\hat{\psi}$ is a null vector of $(\cdot\,,\,\cdot)_{\theta\left(n+\frac{\varphi}{2\pi}\right)}$ for some fixed n. This means that if ψ denotes the symbol of $\hat{\psi}$ then:

$$\psi\left(\vec{q}, e^{ix_1}, \theta\left(n + \frac{\varphi}{2\pi}\right)\right) = 0 \tag{3.12}$$

for all $\vec{q}$ and all e^{ix_1}.

Let ψ_m denote the symbol of $e^{-im\theta\hat{P}_0}\hat{\psi}$. We have:

$$\psi_m\left(\vec{q}, e^{ix_1}, \theta\left(n + \frac{\varphi}{2\pi}\right)\right) = \psi\left(\vec{q}, e^{ix_1}, \theta\left(n + m + \frac{\varphi}{2\pi}\right)\right) . \tag{3.13}$$

Using the fact that $\hat{\psi} \in \mathcal{H}_\theta$ we can write:

$$\psi_m = e^{-i\theta mH}\,\psi , \tag{3.14}$$

where H is the $\theta = 0$ counterpart of $\hat{H}$: $H\psi$ is the symbol of $\hat{H}\hat{\psi}$. We point out that the $\theta = 0$ operator H acts only on the spatial arguments ($\vec{q}$ and e^{ix_1}) of ψ and is time-independent. (This follows from the fact that $\hat{H}$ does not depend on $\hat{P}_0$ and commutes with $\hat{P}_0$.

Using (3.13) and (3.14) we get:

$$\psi\left(\vec{q}, e^{ix_1}, \theta\left(n + m + \frac{\varphi}{2\pi}\right)\right) = e^{-i\theta mH}\,\psi\left(\vec{q}, e^{ix_1}, \theta\left(n + \frac{\varphi}{2\pi}\right)\right) . \tag{3.15}$$

Recalling now that $\hat{\psi}$ is a null vector of $(\cdot\,,\,\cdot)_{\theta\left(n+\frac{\varphi}{2\pi}\right)}$, we find:

$$\psi\left(\vec{q}, e^{ix_1}, \theta\left(n+m+\frac{\varphi}{2\pi}\right)\right) = 0 \tag{3.16}$$

for all $\vec{q}$ and all e^{ix_1}. As m is an arbitrary integer, we see that ψ is identically zero. Consequently the Fourier coefficients $\tilde{\psi}_n(\vec{k}, \omega)$ (see equation (3.1)) are also identically zero. Using (2.13) we finally conclude that $\hat{\psi} = 0$. This shows that the only null vector of (3.4) is the zero vector. This proves that the subspace $\mathcal{H}_\theta$ defined in (3.10) is a Hilbert space.

If $\hat{\psi}$ and $\hat{\phi}$ fulfill (3.9) we have:

$$\begin{aligned}
\left(\hat{\psi}, \hat{\phi}\right)_{\theta\left(m+\frac{\varphi}{2\pi}\right)} &= \left(e^{-i\theta(m-n)\hat{P}_0}\hat{\psi}, e^{-i\theta(m-n)\hat{P}_0}\hat{\phi}\right)_{\theta\left(n+\frac{\varphi}{2\pi}\right)} \\
&= \left(e^{-i\theta(m-n)\hat{H}}\hat{\psi}, e^{-i\theta(m-n)\hat{H}}\hat{\phi}\right)_{\theta\left(n+\frac{\varphi}{2\pi}\right)} \\
&= \left(\hat{\psi}, \hat{\phi}\right)_{\theta\left(n+\frac{\varphi}{2\pi}\right)} ,
\end{aligned} \tag{3.17}$$

which shows that in the subspace defined by (3.9) the family of products (3.4) restricted to $\mathcal{H}_\theta$ collapses to one unique inner product. We denote this product as $\langle\,\cdot\,|\,\cdot\,\rangle$.

Time translations in $\mathcal{H}_\theta$ are implemented by integral powers of the unitary evolution operator $e^{-i\theta\hat{H}}$. The inner product of two elements of $\mathcal{H}_\theta$ is independent of time, i.e.:

$$\langle e^{-i\theta\hat{P}_0}\hat{\psi}\,|\,e^{-i\theta\hat{P}_0}\hat{\phi}\rangle = \langle e^{-i\theta\hat{H}}\hat{\psi}\,|\,e^{-i\theta\hat{H}}\hat{\phi}\rangle = \langle\hat{\psi}\,|\,\hat{\phi}\rangle\,, \tag{3.18}$$

just like in ordinary quantum physics.

Our construction of the Hilbert space of states depends on the choice of a suitable Hamiltonian operator. We remark that in spite of the apparent multiplicity of Hilbert spaces implied by the multiplicity of Hamiltonians, these spaces are in fact equivalent to each other. This can be seen as follows. Let $\hat{H}_1$ and $\hat{H}_2$ be two Hamiltonians fulfilling the necessary conditions for the applicability of the construction described above. Let $\mathcal{H}_\theta^1$ and $\mathcal{H}_\theta^2$ be the Hilbert spaces based on $\hat{H}_1$ and $\hat{H}_2$, respectively. The elements of $\mathcal{H}_\theta^2$ are related to the elements of $\mathcal{H}_\theta^1$ throught the action of the unitary operator

$$\begin{aligned}
&U_{21} := e^{-i\hat{x}_0^R\hat{H}_2}\, e^{i\hat{x}_0^R\hat{H}_1} \\
&U_{21} : \mathcal{H}_\theta^1 \to \mathcal{H}_\theta^2\,,
\end{aligned} \tag{3.19}$$

while an observable $\hat{T}_2$ acting on $\mathcal{H}_\theta^2$ is related to an observable $\hat{T}_1$ acting on $\mathcal{H}_\theta^1$ by conjugation by U_{21}:

$$\hat{T}_2 = U_{21}\,\hat{T}_1\,U_{21}^\dagger\,. \tag{3.20}$$

4. Scattering theory

In this section, we set $Q = \mathbb{R}^D$ and assume the following form for the Hamiltonian acting on $\mathcal{H}_\theta = L^2\left(S^1\right) \otimes L^2(\mathbb{R}^D)$:

$$\hat{H} = \frac{\hat{P}_1^2}{2M} + \frac{\vec{p}^{\,2}}{2M} + V(\,\vec{q}\,) \equiv \hat{H}_0 + V(\,\vec{q}\,)\,. \tag{4.1}$$

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Here $\hat{H}_0$ is the "free" Hamiltonian.

We include no interaction in the noncommutative direction. This is for simplicity. There are still striking physical effects from time discreteness. Further the formalism generalises to more general forms of interaction.

We remark that even though the circle S^1 is compact, there is no difficulty with cluster decomposition and defining asymptotic states if, as we also assume, V falls off sufficiently rapidly as $|\vec{q}| \to \infty$.

Both free and full Hamiltonians $\hat{H}_0$ and $\hat{H}$ play a fundamental role in scattering theory. To each one, we can associate a Hilbert space, but, as we already discussed, they are equivalent to each other. We choose to work uniformly with the Hilbert space $\mathcal{H}^0$ associated with the free Hamiltonian $\hat{H}_0$ in what follows. This implies in particular that $\hat{H}$ will be substituted by its image $\hat{H}'$ under conjugation by the appropriate unitary operator where (see equation (3.20)):

$$\hat{H}' = e^{-i\hat{x}_0^R \hat{H}_0}\, \hat{H}\, e^{i\hat{x}_0^R \hat{H}_0}\,. \tag{4.2}$$

Note that $\hat{H}_0$ is invariant under this conjugation.

Standard scattering theory (cf. refs. [10, 11]) can be adapted readily to the present case. We shall give just the essential details to derive the equation for the scattering matrix. The latter is exact and amenable to approximate solutions by perturbation series or other methods.

Let $|in\,,\,\alpha\rangle$ and $|out\,,\,\alpha\rangle$ denote the *in*- and *out*-states characterised by quantum numbers α. They evolve by $\hat{H}'$ and are prepared at time 0.

By definition, if $|\,in\,,\,\alpha\rangle$ is evolved back to infinite past, it will approach the free state with quantum numbers α evolving by $\hat{H}_0$ (in what follows all limits should be understood in the strong sense):

$$\lim_{\substack{N\to-\infty\\ N\in\mathbb{Z}}} e^{-i\theta N\hat{H}'}|\,in\,,\,\alpha\rangle = \lim_{\substack{N\to-\infty\\ N\in\mathbb{Z}}} e^{-i\theta N\hat{H}_0}|\alpha\rangle\,. \tag{4.3}$$

Thus

$$|in\,,\,\alpha\rangle = \lim_{\substack{N\to-\infty\\ N\in\mathbb{Z}}} \left(e^{i\theta N\hat{H}'}\, e^{-i\theta N\hat{H}_0}\right)|\alpha\rangle =: \Omega^{(+)}|\alpha\rangle\,, \tag{4.4}$$

where

$$\Omega^{(+)} = \lim_{\substack{N\to-\infty\\ N\in\mathbb{Z}}} \left(e^{i\theta N\hat{H}'}\, e^{-i\theta N\hat{H}_0}\right) \tag{4.5}$$

is the Möller operator transforming the free to the *in* state.

Similarly by definition, $|\,out\,,\,\alpha\rangle$ if evolved to infinite future will approach the free state with quantum numbers α evolving by $\hat{H}_0$:

$$\lim_{\substack{N\to+\infty\\ N\in\mathbb{Z}}} e^{-i\theta N\hat{H}'}|\,out\,,\,\alpha\rangle = \lim_{\substack{N\to+\infty\\ N\in\mathbb{Z}}} e^{-i\theta N\hat{H}_0}|\alpha\rangle \tag{4.6}$$

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or

$$
\begin{aligned}
|out\,,\,\alpha\rangle &= \Omega^{(-)}|\alpha\rangle\,, \\
\Omega^{(-)} &= \lim_{\substack{N\to+\infty \\ N\in\mathbb{Z}}} e^{i\theta N\hat{H}'}\,e^{-i\theta N\hat{H}_0}\,.
\end{aligned}
\tag{4.7}
$$

Certain essential features of $\Omega^{(\pm)}$ follow readily from their definitions. Thus,

$$
\begin{aligned}
e^{\pm i\theta\hat{H}'}\,\Omega^{(+)} &= \lim_{\substack{N\to-\infty \\ N\in\mathbb{Z}}} e^{i\theta(N\pm1)\hat{H}'}\,e^{-i\theta N\hat{H}_0} \\
&= \lim_{\substack{N'\to-\infty \\ N\in\mathbb{Z}}} e^{i\theta N'\hat{H}'}\,e^{-i\theta(N'\mp1)\hat{H}_0}
\end{aligned}
\tag{4.8}
$$

or

$$
e^{\pm i\theta\hat{H}'}\,\Omega^{(+)} = \Omega^{(+)}\,e^{\pm i\theta\hat{H}_0}\,, \tag{4.9}
$$

and similarly,

$$
e^{\pm i\theta\hat{H}'}\,\Omega^{(-)} = \Omega^{(-)}\,e^{\pm i\theta\hat{H}_0}\,. \tag{4.10}
$$

The S-matrix element for scattering from an initial to a final state with quantum numbers α and β respectively is

$$
S_{\beta\alpha} = \langle\, out\,,\beta\,|\, in\,,\,\alpha\,\rangle = \langle\beta\,|\,\Omega^{(-)\dagger}\Omega^{(+)}\,|\,\alpha\,\rangle\,, \tag{4.11}
$$

where

$$
S := \Omega^{(-)\dagger}\Omega^{(+)} = \lim_{\substack{M\to+\infty \\ M\in\mathbb{Z}}} U_I(\theta\,,\,M\,,-\infty)\,, \tag{4.12}
$$

$$
U_I(\theta\,,\,M\,,-\infty) = e^{i\theta M\hat{H}_0}e^{-i\theta M\hat{H}'}\,\Omega^{(+)}\,. \tag{4.13}
$$

The operator S is the interaction representation S-matrix.

We next extract the δ-function in $S_{\beta\alpha}$ corresponding to energy conservation $mod\ \frac{2\pi}{\theta}$ as follows. From (4.13),

$$
\begin{aligned}
U_I(\theta\,,\,M+1\,,-\infty) - U_I(\theta\,,\,M\,,-\infty) &= e^{i\theta M\hat{H}_0}\,W\,\Omega^{(+)}\,e^{-i\theta M\hat{H}_0}\,, \\
W &= e^{i\theta\hat{H}_0}\,e^{-i\theta\hat{H}'} - \mathbb{I}\,.
\end{aligned}
\tag{4.14}
$$

Summing both sides from $M = -L$ to $M = J$,

$$
U_I(\theta\,,\,J+1\,,-\infty) - U_I(\theta\,,\,-L\,,-\infty) = \sum_{\substack{k=-L, \\ k\in\mathbb{Z}}}^{J} e^{i\theta k\hat{H}_0}\,W\,\Omega^{(+)}\;e^{-i\theta k\hat{H}_0}\,. \tag{4.15}
$$

Using the boundary conditions

$$
\lim_{\substack{L\to+\infty \\ L\in\mathbb{Z}}} U_I(\theta\,,\,-L\,,-\infty) = \mathbb{I}\,, \tag{4.16}
$$

$$
\lim_{\substack{J\to+\infty \\ J\in\mathbb{Z}}} U_I(\theta\,,\,J+1\,,-\infty) = S\,, \tag{4.17}
$$

we get

$$S = \mathbb{I} + \sum_{\substack{k=-\infty, \\ k\in\mathbb{Z}}}^{+\infty} e^{i\theta k \hat{H}_0}\, W\, \Omega^{(+)}\; e^{-i\theta k \hat{H}_0}\,. \tag{4.18}$$

Let $\alpha\,(\beta)$ denote energy E_α (E_β) and possible additional quantum numbers $\alpha'\,(\beta')$. We write $\alpha = E_\alpha, \alpha'$, $\beta = E_\beta, \beta'$. Then

$$S_{\beta\alpha} \;=\; \delta_{\beta\alpha} - i\theta \left(\sum_{\substack{k=-\infty, \\ k\in\mathbb{Z}}}^{k=+\infty} e^{i\theta k (E_\beta - E_\alpha)} \right) T_{\beta\alpha}\,, \tag{4.19}$$

$$T_{\beta\alpha} \;=\; \frac{i}{\theta} \langle\, E_\beta, \beta' \,|\, W\, \Omega^{(+)} \,|\, E_\alpha, \alpha' \rangle\,. \tag{4.20}$$

Making use of the identity

$$\frac{2\pi}{\theta} \delta_{S^1}(x) = \sum_{n\in\mathbb{Z}} e^{in\theta x}\,, \tag{4.21}$$

where δ_{S^1} is the periodic δ-function with period $\frac{2\pi}{\theta}$, we get the final form of $S_{\beta\alpha}$:

$$S_{\beta\alpha} = \delta_{\beta\alpha} - 2\pi i\, \delta_{S^1}(E_\beta - E_\alpha)\, T_{\beta\alpha}\,. \tag{4.22}$$

Since $T_{\beta\alpha}$ need not be zero if $E_\beta \neq E_\alpha$, this equation clearly shows that scattering conserves energy only mod $\frac{2\pi}{\theta}$.

As $\theta \to 0$, $W \to -i\theta\,(\hat{H}' - \hat{H}_0)$ and $T_{\beta\alpha}$ approaches the familiar form $\langle\, E_\beta, \beta' \,|\, (\hat{H}' - \hat{H}_0)\, \Omega^{(+)} \,|\, E_\alpha, \alpha' \rangle$. In this limit, $\frac{2\pi}{\theta} \to \infty$. Since $(\hat{H}' - \hat{H}_0)\, \Omega^{(+)}$ cannot cause transitions which change energy by an infinite amount, this matrix element approaches zero as $\theta \to 0$ if E_β differs from E_α by a non-zero multiple of $\frac{2\pi}{\theta}$. Thus we may put $E_\beta = E_\alpha$ as well as let $\theta \to 0$ thereby recovering the usual $\theta = 0$ expression for $T_{\beta\alpha}$.

For cross-section calculations, the relevant operator is the restriction of $\frac{i}{\theta}\, W\, \Omega^{(+)}$ to eigenstates of $\hat{H}_0$ for energy eigenvalue E_α. Its overlap with $|E_\beta, \beta'\rangle$ then gives the transition amplitude $T_{\beta\alpha}$. Call this restriction to energy E states of $\hat{H}_0$ as $T(E)$:

$$\begin{aligned} T(E) \;&:=\; \frac{i}{\theta}\, W\, \Omega^{(+)}\, P(E)\,, \\ P(E) \;&=\; \text{projector to energy } E \text{ states}\,. \end{aligned} \tag{4.23}$$

For $\theta \to 0$, $T(E)$ fulfills the Lippman-Schwinger equation. We now derive its analogue for $\theta \neq 0$.

From (4.13), we see that

$$U_I(\theta\,,\, 0\,,\, -\infty) = \Omega^{(+)}\,. \tag{4.24}$$

Hence summing (4.14) from $-\infty$ to -1 and using (4.16) and (4.24), we get

$$\Omega^{(+)}(E + i\epsilon) := \Omega^{(+)}\, P(E) = \left(\mathbb{I} + \sum_{\substack{k=-\infty, \\ k\in\mathbb{Z}}}^{-1} e^{i\theta k(\hat{H}_0 - E - i\epsilon)}\, W\, \Omega^{(+)} \right) P(E)\,, \tag{4.25}$$

where we have put a small positive imaginary part ϵ in E to ensure convergence of the series. (This is standard procedure also for $\theta = 0$. The limit $\epsilon \to 0$ is finally taken.) Summing the series,

$$\Omega^{(+)}(E+i\epsilon) = \left[\mathbb{I} + \frac{1}{e^{i\theta(\hat{H}_0 - E - i\epsilon)} - \mathbb{I}} W\,\Omega^{(+)}(E+i\epsilon)\right] P(E)\,. \tag{4.26}$$

The generalised Lippman-Schwinger equation follows on multiplying (4.26) by $\frac{i}{\theta}\,W$:

$$T(E+i\epsilon) = \left[\frac{i}{\theta}\,W + W\frac{1}{e^{i\theta(\hat{H}_0 - E - i\epsilon)} - \mathbb{I}}\,T(E+i\epsilon)\right] P(E)\,. \tag{4.27}$$

Let

$$W(E) = W\,P(E)\,. \tag{4.28}$$

Then this equation is

$$T(E+i\epsilon) = \frac{i}{\theta}W(E) + W\,\frac{1}{e^{i\theta(\hat{H}_0 - E - i\epsilon)} - \mathbb{I}}\,T(E+i\epsilon)\,. \tag{4.29}$$

Equation (4.29) is well-adapted to expansion of $T(E+i\epsilon)$ in a power series in W. As W is invariant under the substitutions $\hat{H}_0 \to \hat{H}_0 + \frac{2\pi}{\theta}\,n\,,\ \hat{H}' \to \hat{H}' + \frac{2\pi}{\theta}\,m\,,\ n,m \in \mathbb{Z}$, an approximation based on this series is compatible with energy conservation *mod* $\frac{2\pi}{\theta}$.

Let

$$G^{(+)}(E+i\epsilon) = \frac{\theta}{i}\frac{1}{e^{i\theta(\hat{H}_0 - E - i\epsilon)} - \mathbb{I}}\,. \tag{4.30}$$

As $\theta \to 0$,

$$G^{(+)}(E+i\epsilon) \to \frac{1}{E + i\epsilon - \hat{H}_0}\,, \tag{4.31}$$

which is the time-independent Green's function of $\hat{H}_0$. Hence $G^{(+)}(E+i\epsilon)$ is the discrete analogue of this Green's function.

As in standard scattering theory, we can also formally solve (4.29) for $T(E+i\epsilon)$:

$$T(E+i\epsilon) = \frac{1}{\mathbb{I} - \left(\frac{i}{\theta}W\right)\,G^{(+)}(E+i\epsilon)}\left(\frac{i}{\theta}W(E)\right)\,. \tag{4.32}$$

The expansion of the inverse in powers of W gives the perturbation series for $T(E+i\epsilon)$ in W.

The above scattering theory can be applied to a variety of interesting problems. Just as an example, the Born approximation $T_{BA}(E+i\epsilon)$ for the Yukawa potential

$$V(r) = V_0\,\frac{e^{-r/a}}{r} \tag{4.33}$$

reads

$$T_{BA}(E+i\epsilon) = \frac{i}{\theta}\left(e^{i\theta\hat{H}_0}e^{-i\theta\left(\hat{H}_0 + V_0\,\frac{e^{-r/a}}{r}\right)} - \mathbb{I}\right) P(E)\,. \tag{4.34}$$

The matrix element of $T_{BA}(E_\alpha + i\epsilon)$ between an initial free state with energy E_α and a final state with energy $E_\beta \neq E_\alpha$ gives the transition matrix $T_{\beta\alpha}$. As long as the Yukawa potential does not commute with the free Hamiltonian $\hat{H}_0$, $T_{\beta\alpha}$ is nontrivial, the Yukawa interaction being responsible for transitions between states with different energies. The observable transitions are the scattering channels allowed by the periodic delta function $\delta_{S^1}(E_\beta - E_\alpha)$ in (4.22).

It should be noted that just as in the ordinary case, the parameter V_0 in (4.33) controls the size of the perturbation $\frac{i}{\theta}W$.

5. Conclusions

The present paper shows explicitly that scattering processes with discrete time evolution lead to energy nonconservation. Specifically, energy is conserved only *mod* $\frac{2\pi}{\theta}$, θ being the elementary time interval.

The time independent scattering theory developed in section 4 allows perturbation calculations, just like in ordinary quantum theory. This fact suggests a variety of interesting applications of this formalism.

In addition to the perturbative approach, it will be interesting to construct exactly soluble models in the future, in order to give us detailed insights on the general behaviour of the transition matrix.

The results of this paper have the merit of permiting straightforward calculations of transition rates. This fact will certainly lead to striking phenomenological consequences which can be confronted with experiments.

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References

[1] M. Chaichian, A. Demichev, P. Prešnajder and A. Tureanu, *Space-time noncommutativity, discreteness of time and unitarity*, *Eur. Phys. J.* **C 20** (2001) 767 [hep-th/0007156].

[2] A.P. Balachandran, T.R. Govindarajan, A.G. Martins and P. Teotonio-Sobrinho, *Time-space noncommutativity: quantised evolutions*, *JHEP* **11** (2004) 068 [hep-th/0410067].

[3] J. Gomis and T. Mehen, *Space-time noncommutative field theories and unitarity*, *Nucl. Phys.* **B 591** (2000) 265 [hep-th/0005129].

[4] D. Bahns, *Unitary quantum field theory on the noncommutative Minkowski space*, *Fortschr. Phys.* **51** (2003) 658 [hep-th/0212266].

[5] S. Doplicher, K. Fredenhagen and J.E. Roberts, *Space-time quantization induced by classical gravity*, *Phys. Lett.* **B 331** (1994) 39; *The quantum structure of space-time at the Planck scale and quantum fields*, *Commun. Math. Phys.* **172** (1995) 187 [hep-th/0303037].

[6] A.P. Balachandran, T.R. Govindarajan, C. Molina and P. Teotonio-Sobrinho, *Unitary quantum physics with time-space noncommutativity*, *JHEP* **10** (2004) 072 [hep-th/0406125].

[7] A.P. Balachandran and L. Chandar, *Discrete time from quantum physics*, *Nucl. Phys.* **B 428** (1994) 435 [hep-th/9404193].

[8] H.-J. Matschull and M. Welling, *Quantum mechanics of a point particle in* $2+1$ *dimensional gravity*, *Class. and Quant. Grav.* **15** (1998) 2981 [gr-qc/9708054].

[9] G. 't Hooft, *Canonical quantization of gravitating point particles in* $(2+1)$*-dimensions*, *Class. and Quant. Grav.* **10** (1993) 1653 [gr-qc/9305008]; *Quantization of point particles in 2+1 dimensional gravity and space-time discreteness*, *Class. and Quant. Grav.* **13** (1996) 1023 [gr-qc/9601014].

[10] R.G. Newton, *Scattering theory of waves and particles*, McGraw-Hill Book Company (1966).

[11] M.L. Goldberger and K.M. Watson, *Collision theory*, John Wiley and Sons, Inc. (1964).

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